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Reduced-Order H_∞ Compensator Design for an Aircraft Control Problem

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A recently introduced method for designing H_∞ compensators based on minimal-order observers is considered for an aircraft control problem. The purpose of this paper is to bridge the gap between theory and application by presenting a practical utilization of a new design approach. Manual flight control systems for the lateral axis of a fighter aircraft are developed using both full-order and reduced-order compensators, and the results are compared. It is demonstrated that this method can be used to directly design reduced-order compensators that result in a system satisfying a closed-loop H_∞ bound.

I. Introduction

THE H_∞ compensator design has been recognized as one of the effective tools for robust control. Many control problems in diverse areas of application can be placed in an H_∞ framework. It is well known that full-order H_∞ compensators can be designed using various approaches. The issue of reducing H_∞ compensator complexity has attracted considerable attention in the past few years. In many applications it is desirable to reduce the order of the compensator if the resulting loss in performance is small enough. Hsu et al.[1] have used the bounded real lemma to derive proper H_∞ compensators based on minimal-order observers. Design equations have been derived whose solutions provide a sufficient condition to satisfy a closed-loop H_∞ norm bound.[1]

The purpose of this paper is to present an application of this design method to an aircraft control problem. The focus of this paper is to show how the method is applied in a relatively straightforward and practical design process. Although performance and stability robustness measures based on uncertainty modeling could be included in the design, they are not explicitly included here to simplify the discussion. The design problem addressed is a manual flight control system for the lateral axis of a fighter aircraft. The design is based on one done by Adams,[2] who uses pseudocontrol and an inner/outer-loop structure to achieve robustness and performance across the flight envelope. Only the outer-loop design to achieve flying qualities requirements is addressed here.

II. Design Methodology

In this paper, we consider the following linear control problem. We are given a system described by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{1}$$

where x, y, z, u, and w are the state, the measured output, the controlled output, the control input, and the disturbance input, respectively. The dimensions of x, y, z, u, and w are n, p, r, m, and q, respectively. We assume $\{A, B_2, C_2\}$ is stabilizable and detectable, D_{12} is of full column rank, and D_{21} is of full row rank. We want to find a controller such that the closed-loop system is internally stable and the H_∞ norm from the disturbance input to the controlled output, $\|T_{zw}\|_\infty$ is less than a positive number γ. A controller of order n exists if and only if the unique stabilizing solutions to two algebraic Riccati equations are positive semidefinite and the spectral radius of the product is less than γ^2, as described by Doyle et al.[3] and Glover and Doyle.[4]

Hsu et al.[1] have derived design equations that provide a sufficient condition for such a controller of order n-p to exist. A proper observer-based controller of order n-py

$$\begin{aligned} \dot{\xi} &= F\xi + G_1 y + G_2 u \\ u &= -H\xi - Jy \end{aligned} \tag{2}$$

that satisfies the two closed-loop specifications is said to be admissible. The closed-loop system can be obtained by combining Eqs. (1) and (2)

$$\begin{aligned} \dot{x}_{cl} &= A_{cl} x_{cl} + B_{cl} w \\ z &= C_{cl} x_{cl} + D_{cl} w \end{aligned} \tag{3}$$

where

$$x_{cl} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \qquad A_{cl} = \begin{bmatrix} A - B_2 J C_2 & -B_2 H \\ (G_1 - G_2 J) C_2 & F - G_2 H \end{bmatrix}$$

$$B_{cl} = \begin{bmatrix} B_1 - B_2 J D_{21} \\ (G_1 - G_2 J) D_{21} \end{bmatrix}, \qquad D_{cl} = -D_{12} J D_{21} \tag{4}$$

$$C_{cl} = [C_1 - D_{12} J C_2 \quad -D_{12} H]$$

Theorem 1 in Ref. 1 states that the controller of order n-p as defined in Eq. (2) is an admissible compensator for the plant (1) if there exist positive-semidefinite matrices P_1 and $\bar{Q}_2$ satisfying Riccati inequalities[1]

$$\begin{aligned} A^T P_1 + P_1 A + C_1^T C_1 - (P_1 B_2 + C_1^T D_{12})(D_{12}^T D_{12})^{-1} \\ \times (B_2^T P_1 + D_{12}^T C_1) + \gamma^{-2} P_1 B_1 B_1^T P_1 \le 0 \end{aligned} \tag{5}$$

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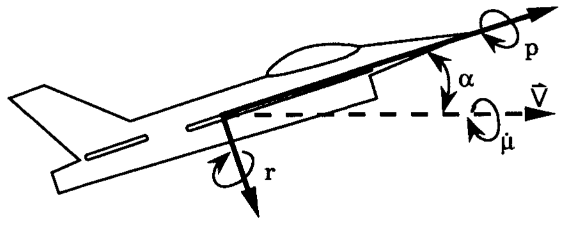

Fig. 1 State and output variables.

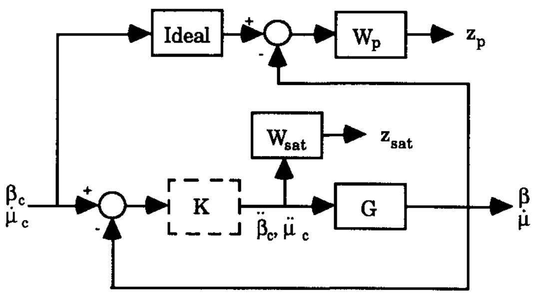

Fig. 2 Design model.

$$\bar{Q}_2\bar{A}^T + \bar{A}\bar{Q}_2 + B_1B_1^T + \gamma^{-2}\bar{Q}_2\bar{K}^T(D_{12}^TD_{12})\bar{K}\bar{Q}_2 - (\bar{Q}_2\bar{C}_2^T + B_1D_{21}^T)(D_{21}D_{21}^T)^{-1}(\bar{C}_2\bar{Q}_2 + D_{21}B_1^T) \le 0 \tag{6}$$

where

$$\begin{aligned} \bar{A} &= A + \gamma^{-2}B_1B_1^TP_1 \\ \bar{C}_2 &= C_2 + \gamma^{-2}D_{21}B_1^TP_1 \\ \bar{K} &= (D_{12}^TD_{12})^{-1}(B_2^TP_1 + D_{12}^TC_1) \end{aligned} \tag{7}$$

and the controller parameters are given as

$$\begin{aligned} V\tilde{A} &= FV \\ G_1 &= V(\bar{Q}_2\bar{C}_2^T + B_1D_{21}^T)(D_{21}D_{21}^T)^{-1} \\ G_2 &= V\{B_2 + \gamma^{-2}\bar{Q}_2(P_1B_2 + C_1^TD_{12})\} \\ [J \quad H] &= \bar{K}\begin{bmatrix} \bar{C}_2 \\ V \end{bmatrix}^{-1} \end{aligned} \tag{8}$$

where

$$\tilde{A} = \bar{A} + \gamma^{-2}\bar{Q}_2\bar{K}^TD_{12}^TD_{12}\bar{K} - (\bar{Q}_2\bar{C}_2^T + B_1D_{21}^T)(D_{21}D_{21}^T)^{-1}\bar{C}_2 \tag{9}$$

and the resulting controller must also satisfy

$$\Omega \equiv \gamma^2 I - D_{cl}^TD_{cl} > 0 \tag{10}$$

This theorem provides the following compensator design procedure:

Step 1: Solve Eq. (5) as a Riccati equality to find P_1 and then solve Eq. (6) as a Riccati equality to find $\bar{Q}_2$.

Step 2: Compute $\tilde{A}$ and select F to be a block diagonal matrix with eigenvalues that are a subset of the eigenvalues of $\tilde{A}$. Solve the Sylvester equation $V\tilde{A} = FV$ for V.

Step 3: Compute design parameters G_1, G_2, J, and H using system (8).

Step 4: Check the condition in Eq. (10); if it is not satisfied, increase γ and go to step 1.

The two Riccati equations in step 1 are not mutually coupled, so their solutions can be easily obtained. The solution to the homogeneous Sylvester equation in step 2 is not unique. F is chosen to be any n-p submatrix of the Jordan form of $\tilde{A}$ that satisfies condition (10). Step 4 is needed to assure that the H_∞ norm constraint is satisfied. A search over γ must be performed, as in standard H_∞ formulations, to find the minimal H_∞ norm solution. Finally, the standard full-order strictly proper H_∞ central controller[3,4] can be obtained by simply letting $V = I$, $G_2 = B_2$, and $J = 0$. The resulting full-order compensator is the same as the central controller with P_1 and $\bar{Q}_2$ corresponding to X_∞ and $Z_\infty Y_\infty$ (Ref. 3) of the central controller. A proof of this is presented by Hsu et al.[1] in Appendix B of their paper.

III. Problem Formulation

We consider the design of a manual flight control system for the lateral axis of the single engine VISTA F-16 supersonic test vehicle. This aircraft has leading and trailing edge flaps, an all movable horizontal tail, and a single rudder. The horizontal tail is used symmetrically for pitch control and differentially for roll control. The pilot controls include a force-feel side stick, rudder pedals, and a throttle. The actual control inputs used in this problem are asymmetric horizontal tail, asymmetric flaps, and rudder. As described by Adams,[2] these are transformed into two generalized inputs: sideslip acceleration ($\ddot\beta$) and stability axis roll acceleration ($\ddot\mu$) commands. A form of dynamic inversion is applied to the transformed system to equalize the system over a range of dynamic pressures. The resulting central system is chosen such that the error between the central model and all other models is relatively small over the flight envelope. By equalizing the plant dynamics across the flight envelope, gain scheduling is avoided and a single outer-loop design can be performed. The equalized plant is described by

$$A = \begin{bmatrix} -0.1689 & 0.0759 & -0.9952 \\ -26.859 & -2.5472 & 0.0689 \\ 9.3603 & -0.1773 & -2.4792 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0 & 0.9971 & 0.0755 \end{bmatrix} \tag{11}$$

The state, control, and output vectors are given by

$$x = [\beta \quad p \quad r]^T \qquad u = [\ddot\beta \quad \ddot\mu]^T \qquad y = [\beta \quad \dot\mu]^T \tag{12}$$

The states, inputs, and outputs are, in order: sideslip, body axis roll rate, yaw rate, sideslip acceleration, stability axis roll acceleration, sideslip, and stability axis roll rate. Figure 1 shows the difference between body axis and stability axis roll rates, where α is the angle of attack and V is the velocity vector.

The primary measure of manual flight control performance is adherence to flying quality requirements. These requirements are given in terms of low-order transfer functions that approximate the desired dynamics between pilot input and aircraft response over the frequency range considered important.[2] The lateral stick and pedal commands from the pilot translate into a sideslip angle command and a stability axis roll rate command. Ideal models of the desired aircraft response to transformed pilot inputs are given by

$$\frac{\beta}{\beta_c} = \frac{\omega_D^2}{s^2 + 2\zeta_D\omega_D s + \omega_D^2} \qquad \frac{\dot\mu}{\dot\mu_c} = \frac{1/T_R}{s + 1/T_R} \tag{13}$$

where $\omega_D = 3.0$ rad/s, $\zeta_D = 0.71$, and $T_R = 0.33$. An H_∞ optimization problem can then be formulated directly in terms of minimizing the maximum error between the ideal models given in Eqs. (13) and the actual transfer functions. Figure 2 shows the structure of the design model, where Ideal, G, and K are the ideal models, the equalized plant, and the controller to be designed.

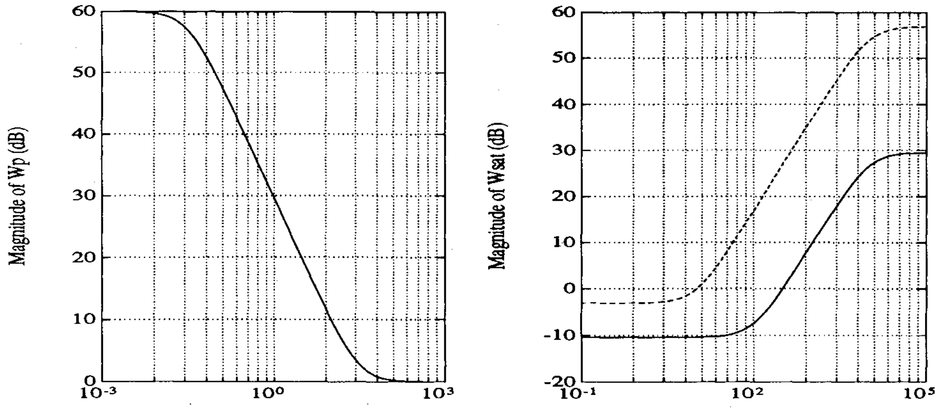

Fig. 3 W_p and W_{sat} (solid is design 1, dashed is design 2).

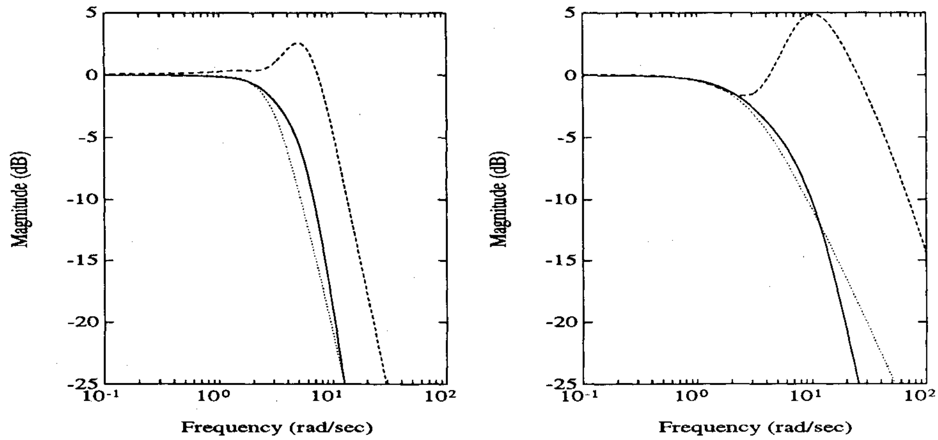

Fig. 4 β/β_c and $\dot{\mu}/\dot{\mu}_c$ for design 1.

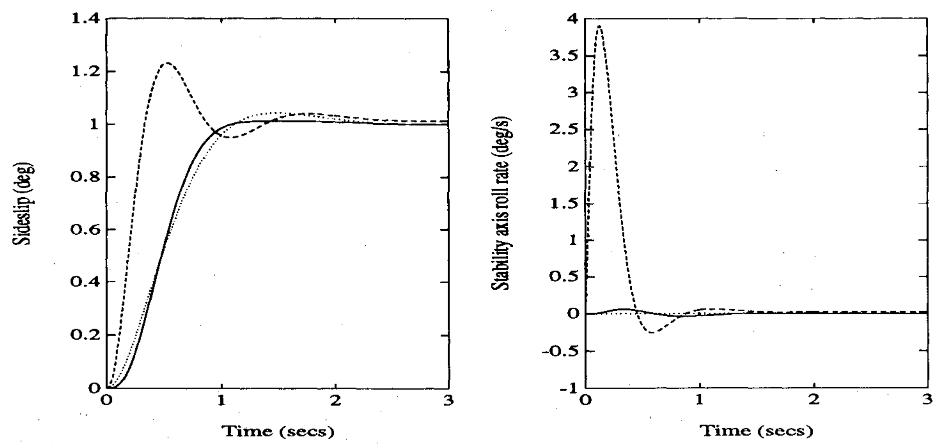

Fig. 5 Response to sideslip step command for design 1.

The performance weight W_p is formulated to keep the error between the ideal model and the complementary sensitivity function small at frequencies less than 10 rad/s. The saturation weight W_{sat} is formulated to keep the control effort within reasonable bounds.

IV. Designs

Two designs were done with different values for W_{sat} that illustrate some of the advantages and difficulties of directly designing the reduced-order compensator. In the first design, the full-order compensator gave very good matching, whereas the reduced-order compensator did not. In the second design, the full-order compensator did not match quite as well, but the reduced-order compensator performed much better than in the first design. The performance weighting used was

$$W_p = \frac{s + 30}{s + 0.03} I_{2\times 2} \tag{14}$$

The control weightings used were

$$W_{sat1} = 30\frac{s + 100}{s + 10000} I_{2\times 2} \qquad W_{sat2} = 700\frac{s + 10}{s + 10000} I_{2\times 2} \tag{15}$$

Figure 3 shows magnitude plots for these weighting functions.

The eigenvalues of $\tilde{A}$ for both designs are: $\lambda_{1,2} = -0.03$, $\lambda_3 = -3.0$, $\lambda_{4,5} = -1.1437 \pm 3.2515i$, $\lambda_6 = -2.9079$, $\lambda_{7,8}$

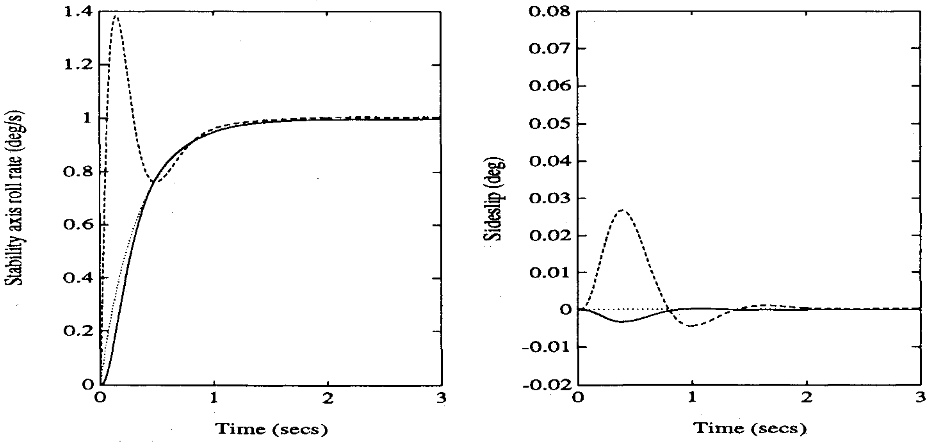

Fig. 6 Response to stability axis roll rate step command for design 1.

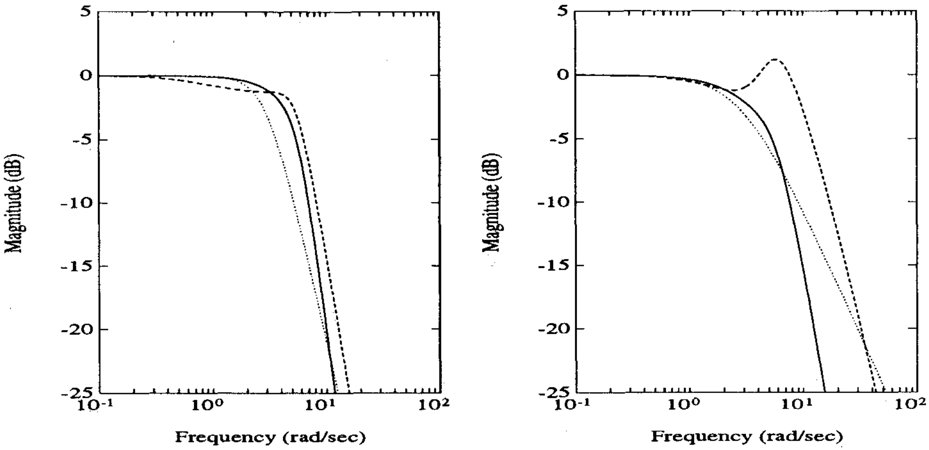

Fig. 7 β/β_c and $\dot{\mu}/\dot{\mu}_c$ for design 2.

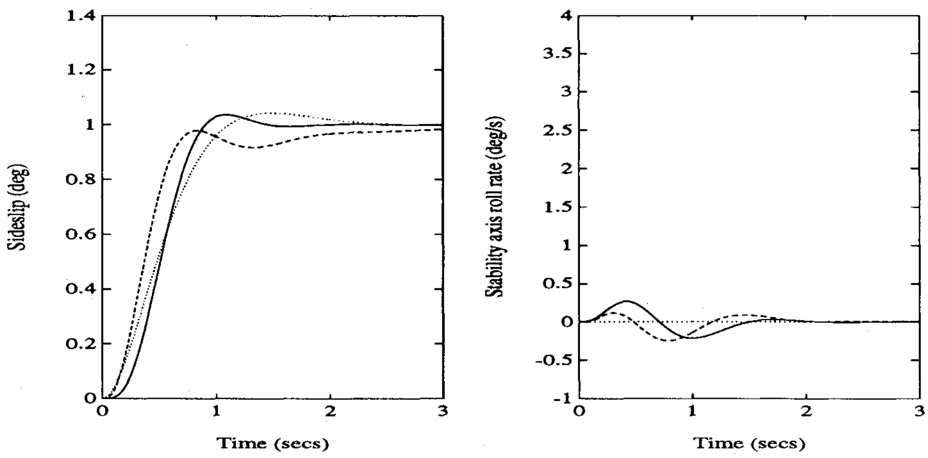

Fig. 8 Response to sideslip step command for design 2.

$= -2.1300 \pm 2.1126i$, and $\lambda_{9,10} = -10{,}000$. The matrix F that satisfies condition (10) is block diagonal with the eigenvalues $\lambda_{1,2}$, λ_3, λ_6, $\lambda_{7,8}$, and $\lambda_{9,10}$.

Figure 4 shows the transfer functions β/β_c and $\dot{\mu}/\dot{\mu}_c$ for the ideal model (dotted line), the full-order compensator (solid line), and the reduced-order compensator (dashed line). The line markings in subsequent figures correspond to those of Fig. 4. Time responses are shown in Figs. 5 and 6. The system with the full-order compensator tracks the desired response well, with almost no cross-coupling between step responses. The peaking in the frequency responses for the system with the reduced-order compensator in the region of 10 rad/s causes significant overshoot in the step responses. There is also significant cross-coupling in the sideslip step response.

Since we are using a direct design technique to find a reduced-order compensator, we can see how to adjust the weights to achieve better responses with the reduced-order compensator. From Fig. 4, most of the unwanted peaking is in the range of 10 to 100 rad/s. Figure 3 shows that the control effort weighting was increased significantly in this region, as well as at lower frequencies. The frequency responses obtained with the new weightings are shown in Fig. 7. Although the

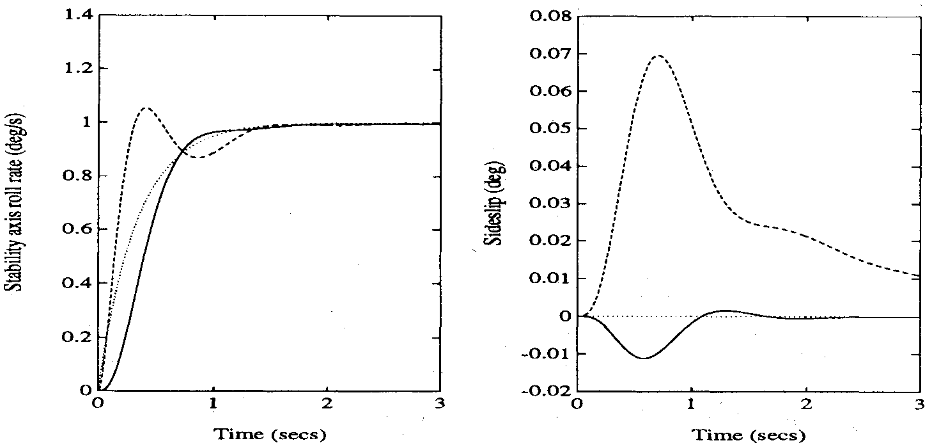

Fig. 9 Response to stability axis roll rate step command for design 2.

Table 1 Compensator eigenvalues and values for $\gamma = \|T_{zw}\|_\infty$

Design 1		Design 2	
Full-order $\gamma = 8.62$	Reduced-order $\gamma = 29.8$	Full-order $\gamma = 20.1$	Reduced-order $\gamma = 20.1$
-100.01	-123.7	-12.09	-18.52
-99.54	-99.49	$-8.111 \pm 4.930i$	$-8.129 \pm 4.689i$
$-9.122 \pm 5.067i$	-9.482	$-4.179 \pm 6.442i$	$-2.127 \pm 2.222i$
$-5.000 \pm 5.585i$	-0.9801	-6.712	-1.570
$-2.676 \pm 2.261i$	3.0244	$-2.222 \pm 2.133i$	-0.001
-0.0027	1.3261	-0.007	0.001
-0.0007	1.0320	-0.002	—
—	0.0302	—	—

frequency response is slightly lower, below 2 rad/s where good matching is desired, this is made up for by the significant reduction in peaking in the range of 10 rad/s.

The step responses in Figs. 8 and 9 show slightly worse responses for the full-order case, but significantly improved performance for the reduced-order system. The overshoots were nearly eliminated, as was the sideslip command cross-coupling. Although the roll rate command cross-coupling increased, the values are still reasonable.

An interesting property of the second design (see Table 1) is that although it achieves the same value for γ with the reduced-order compensator as with the full-order compensator, the closed-loop response is different. This difference is probably due to the fact that the reduced-order compensator is proper, whereas the full-order compensator is strictly proper. In the proper case, the resulting closed-loop maximum singular value is constant at high frequencies, whereas in the strictly proper case, the maximum singular value rolls off at higher frequencies. The result of this property of the H_∞ solution is that the shape of the weights at high frequencies can affect the solution found at lower frequencies, even though the designer is only looking for sufficient rolloff at high frequencies. The flexibility to shape these high-frequency weightings is limited by the desire to use the lowest order possible for the weighting functions. This result implies that using proper instead of strictly proper compensators will generally give solutions of a different form.

Although the reduced-order design technique guarantees closed-loop stability, Table 1 shows that the reduced-order compensator itself may be unstable, a trait that is generally undesirable.[5] Other examples have produced stable reduced-order compensators, indicating that this property is problem dependent. Other methods of finding reduced-order controllers may also produce either stable or unstable controllers, depending on the problem.[6] When solving for full-order H_∞ compensators, the same problem can occur.[7,8] As γ is increased, the set of admissible controllers becomes larger, so that it becomes more likely a stable compensator can be found. Veillette et al.[9] have derived a sufficient condition for finding an admissible full-order controller that is also stable, at the cost of increasing γ. In general, because of its importance in practical applications, the theory of finding stable H_∞ compensators for strongly stabilizable systems needs to be further explored.

V. Conclusions

A new method developed for designing H_∞ compensators based on minimal-order observers was used to design a reduced-order manual flight control system for the lateral axis of a fighter aircraft. Although the technique only provides a sufficient condition for the existence of a controller satisfying stability and H_∞-norm constraints, the approach is demonstrated to be useful in a practical problem. It was shown that the method allows the designer to approach the reduced-order control problem in the same manner as with standard H_∞ design techniques in the sense of using weightings to make tradeoffs over different frequency ranges. Some differences result from the fact that the resulting controller is proper rather than strictly proper.

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